ETMAG

LECTURE 13

- The dimension of a vector space
- Matrices
- The rank of a matrix
- Row operations

Theorem. (Last lecture)

Let V be a vector space over a field $\mathbb{K}$. A set $S = \{v_1, v_2, \dots , v_n\} \subseteq$ V is linearly independent iff no vector from S is a linear combination of the remaining $n - 1$ vectors.

In other words:

Remark.

S $= \{v_1, v_2, \dots , vn\}$ is linearly independent iff

$$(\forall i = 1,2, \dots, n)(v_i \notin span(S \setminus \{v_i\}).$$

Examples.

1. In every $\mathbb{K}^n$ the set of *unit vectors*, $\{(1,0, \dots, 0), (0,1,0, \dots, 0), \dots, (0,0, \dots, 0,1)\}$ is linearly independent.
   $a_1(1,0, \dots, 0) + a_2(0,1,0, \dots, 0) + \dots + a_n(0,0, \dots, 0,1) = (a_1, a_2, \dots, a_n) = (0,0, \dots, 0)$ only if $a_1 = a_2 = \cdots a_n = 0$.

3. Is the set $\{1, x, x^2, \dots, x^n\}$ a linearly independent subset of $\mathbb{R}[x]$?
(induction on n)
$\{1\}$ and $\{1, x\}$ are linearly independent. Suppose $\{1, x, x^2, \dots, x^n\}$ is linearly independent and $\{1, x, x^2, \dots, x^{n+1}\}$ is not. This implies that $x^{n+1} = \sum_{s=0}^{n} a_s x^s$
Differentiating both sides $n+1$ times yields $(n+1)! = 0$, a contradiction.

Definition.

If a set $S, S \subseteq \mathrm{V}$, is linearly independent and $span(S) = V$ then S is called *a basis* of V.

Remark.

We defined linear independence only for finite sets of vectors. It can be extended to infinite sets but in this course we will only consider vector spaces which have finite bases. Such spaces are said to be *"finite-dimensional*".

Examples.

1. In every $\mathbb{K}^n$ the set of *unit vectors*, $S = \{(1,0, \dots, 0),$ $(0,1,0, \dots, 0), \dots, (0,0, \dots, 0,1)\}$ is a basis.
   We know it is linearly independent. Obviously, for every vector $v = (x_1, x_2, \dots, x_n)$ we can write $v = x_1(1,0, \dots, 0) + x_2(0,1,0, \dots, 0), + \ \dots + x_n(0,0, \dots, 0,1)$ so
$$span(S) = V.$$
2. $\{1, x, x^2, \dots, x^n\}$ is a basis for $\mathbb{R}_n[x]$.
3. $\mathbb{R}[x]$ has no finite basis.
4. $\{1, i\}$ is a basis for $\mathbb{C}$ over $\mathbb{R}$.
5. $\{1\}, \{i\}, \{1 + i\}$ are bases for $\mathbb{C}$ over $\mathbb{C}$.

Theorem

A set $S = \{v_1, v_2, \dots, v_n\}$ is a basis of a vector space V over $\mathbb{K}$ iff for every $v \in$ V there exist unique coefficients $a_1, a_2, \dots, a_n$ such that $v = a_1 v_1 + a_2 v_2 + \ \dots + a_n v_n = \sum_{s=1}^{n} a_s v_s$

Proof. (Left as an exercise)

Theorem.

If S and R are bases of a (finite dimensional) vector space v then $|S| = |R|$.

We skip the proof.

Definition.

If V is a (finite dimensional) vector space then the *dimension of* V is the number of vectors in any basis of V.

Examples …

Theorem. (6-pack theorem, properties of bases)
Suppose V is a vector space, $\dim(V) = n$, $n > 0$ and $S \subseteq V$.
Then

1. If $|S| = n$ and S is linearly independent, then S is a basis for V
2. If $|S| = n$ and $span(S) = V$ then S is a basis for V
3. If S is linearly independent, then S is a subset of a basis of V
4. If $span(S) = V$ then S contains a basis of V
5. S is a basis of V iff S is a maximal linearly independent subset of V
6. S is a basis of V iff S is a minimal spanning set for V.

MATRICES

Definition.

An $m \times n$ *matrix* over a field $\mathbb{F}$ is a $m \times n$ ("m by n") array of elements of the field (usually numbers). The horizontal lines of the array are referred to as *rows* and the vertical ones as *columns* of the matrix. The individual elements are called *entries* of the matrix.

Thus, an $m \times n$ matrix has m rows, n columns and mn entries.

If $m = n$ we call A a *square matrix*.

The set of all $m \times n$ matrices over a field $\mathbb{F}$ is denoted by $\mathbb{F}^{m \times n}$.

Matrices will be denoted by capital letters and their entries by the corresponding lower-case letters. Thus, in case of a matrix A we will write $A(i,j) = a_{i,j}$ and will refer to $a_{i,j}$ as the element of the i-th row and j-th column of A.

On the other hand, we will use the symbol $[a_{i,j}]$ to denote the matrix A with entries $a_{i,j}$.

Rows and columns of a matrix are vectors from $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively, and will be denoted by $r_1, r_2, \ldots, r_m$ and $c_1, c_2, \ldots, c_n$. The expression $m \times n$ is called the *size* of the matrix.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Algebra of matrices

Definition.

Matrix addition is only defined for matrices of matching sizes,

$(A+B)(i,j) = A(i,j) + B(i,j)$ for every $i, j,\ \ 1 \leq i \leq m, 1 \leq j \leq n$.

$(cA)(i,j) = cA(i,j), 1 \leq i \leq m, 1 \leq j \leq n$ (multiplication of a function by a constant)

Matrix multiplication.

Definition.
Let A be an $m \times n$ and B an $n \times q$ matrix. Then
$(AB)(i,j) = \sum_{s=1}^{n} A(i,s)B(s,j)$, for every $1 \leq i \leq m$ and $1 \leq j \leq q$.

- If the number of columns of A differs from the number of rows of B, AB is not defined.
- AB is clearly an $m \times q$ matrix.

Matrix multiplication is non-commutative, it may even happen that AB exists while BA does not.

Example (Matrix multiplication).

1. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 3 & 1 \\ 1 & -2 & 2 \end{bmatrix}$. Then,

$$AB = \begin{bmatrix} 2+1+2 & 0-3-4 & 2-1+4 \\ 4+0-3 & 0+0+6 & 4+0-6 \end{bmatrix} = \begin{bmatrix} 5 & -7 & 5 \\ 1 & 6 & -2 \end{bmatrix}.$$

2. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

The second example proves that AB may differ from BA even when both products exist and have the same size.

Example. (Multiplication trick).

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 0 & 3 \end{bmatrix} B$$

2 4 0

$$A \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 3 \end{bmatrix} A$$

$$B \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$$

$$X \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

Arithmetic properties of matrices

Theorem.

1. $(\mathbb{F}^{m\times n}, +)$ is a vector space over $\mathbb{F}$.
2. Matrix multiplication is associative but, in general, not commutative.
3. The $n\times n$ matrix I defined as $I(s,t) = \begin{cases} 1 \ if \ \ s = t \\ 0 \ if \ \ s \neq t \end{cases}$ is the identity element for matrix multiplication in $\mathbb{F}^{n\times n}$:

$$\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1a + 0p + 0x & \ldots & \ldots \\ 0a + 1p + 0x & \ldots & \ldots \\ 0a + 0p + 1x & \ldots & \ldots \end{bmatrix}$$

4. Matrix multiplication is distributive over matrix addition:

$$A(B + C) = AB + AC$$

Definition.

If A is an $m \times n$ matrix then A *transposed* is the $n \times m$ matrix A^{T} such that for each i and j ($1 \le i \le n, 1 \le j \le m$) $A^T(i, j) = A(j, i)$.

Example. $\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 2 & 3 \end{bmatrix}$

Definition.

If $A = AT$ then A is said to be *symmetric*.

Fact. (obvious)

For every matrix A, $(A^T)^T = A$

For every two matrices of matching sizes, $(A+B)^T = A^T + B^T$.

For every two A and B such that AB exists, $(AB)^T = B^T A^T$.

Proof (of the last statement).

$$\sum_{s=1}^{n} A(i,s)B(s,j) = \sum_{s=1}^{n} A^T(s,i)B^T(j,s) = \sum_{s=1}^{n} B^T(j,s)A^T(s,i) =$$

$B^T A^T(j,i)$. QED

Definition.

Let A be an $n \times k$ matrix. We say that A is a *row echelon* matrix iff for every $i = 2, 3, \ldots, n$

(a) if r_i is a nonzero row of A then r_{i-1} is also a nonzero row,

(b) if $a_{i,j}$ is the first nonzero entry in r_i and $a_{i-1,p}$ is the first nonzero entry in r_{i-1} then $p < j$

If, in addition,

(c) the first nonzero entry in each nonzero row is equal to 1

(d) the first nonzero entry in each nonzero row is the only nonzero entry in its column

then A is called a *row canonical* matrix.

Example.

$$A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 7 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition.

The following transformations of a matrix are called *elementary row operations (EROS):*

1. $r_i \leftrightarrow r_j$ - replacing row r_i with r_j and vice versa (row swapping)
2. $r_i \leftarrow cr_i$ - replacing row r_i with r_i scaled by a nonzero constant c. In practice, we abbreviate the symbol to cr_i
3. $r_i \leftarrow r_i + r_j$ - replacing row r_i with the sum of r_i and r_j (adding of r_j to r_i). Usually, we write simply $r_i + r_j$.
4. $r_i \leftarrow r_i + cr_j$ - replacing row r_i with the sum of r_i and the multiple of r_j by a constant c. We just write $r_i + cr_j$ for short.

Notice that 4 is a composition of 2 and 3. Namely, we do cr_j, then $r_i + r_j$ (here r_i denotes the "new" row *j*, after scaling) and finally $c^{-1}r_j$ to convert row *j* to its original form.

Definition.

Matrices A and B are said to be *row-equivalent* iff A can be transformed into B by a (finite) number of elementary row operations. We denote row-equivalence by $A \sim B$.

Proposition.

The relation of row-equivalence is an equivalence relation on $\mathbb{F}^{n \times m}$.

Theorem.

Every matrix is row equivalent to a row-canonical matrix.

(Every matrix can be row-reduced to a row canonical one)

(Every equivalence class of $\sim$ contains a row-canonical matrix)

Example.

$$A=\begin{bmatrix}0&2&1&1\\1&1&1&0\\2&1&0&0\\2&6&4&2\end{bmatrix} r_1\leftrightarrow r_4 \begin{bmatrix}2&6&4&2\\1&1&1&0\\2&1&0&0\\0&2&1&1\end{bmatrix} \frac{r_1}{2} \begin{bmatrix}1&3&2&1\\1&1&1&0\\2&1&0&0\\0&2&1&1\end{bmatrix}$$

$$r_2-r_1, r_3-2r_1 \begin{bmatrix}1&3&2&1\\0&-2&-1&-1\\0&-5&-4&-2\\0&2&1&1\end{bmatrix} 2r_2-r_3 \begin{bmatrix}1&3&2&1\\0&1&2&0\\0&-5&-4&-2\\0&2&1&1\end{bmatrix}$$

$$r_3+5r_2, r_4-2r_2 \begin{bmatrix}1&3&2&1\\0&1&2&0\\0&0&6&-2\\0&0&-3&1\end{bmatrix} r_4+\frac{1}{2}r_3, \frac{1}{6}r_3 \begin{bmatrix}1&3&2&1\\0&1&2&0\\0&0&1&-\frac{1}{3}\\0&0&0&0\end{bmatrix}$$

$$r_1-3r_2, r_2-2r_3 \begin{bmatrix}1&0&-4&1\\0&1&0&\frac{2}{3}\\0&0&1&-\frac{1}{3}\\0&0&0&0\end{bmatrix} r_1+4r_3 \begin{bmatrix}1&0&0&-\frac{1}{3}\\0&1&0&\frac{2}{3}\\0&0&1&-\frac{1}{3}\\0&0&0&0\end{bmatrix}$$

Definition.

The *row rank* of an $n \times m$ matrix A, $r(A)$, is the dimension of the subspace of $\mathbb{F}^m$ spanned by rows of A.

Theorem.

For every two matrices A and B, if $A \sim B$ then $r(A) = r(B)$.

Proof. (skipped)

Note. Since the rank of any row echelon matrix is clearly the number of its nonzero rows, the theorem provides a method for calculating the rank of the matrix - row reduce the matrix to a row echelon one and count its nonzero rows.

Theorem.

For every matrix A, $r(A) = r(A^T)$.

We skip the proof .